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Converse Theorems of Convexity for Bernstein Polynomials over Triangles

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Let $B^n(f; P)$ be the nth Bernstein polynomial of a real function $f(P)$ whose domain is a triangle T. We show in this paper that if $f(P)$ is continuous on T and one of the inequalities $B^n(f; P) \geqslant f(P)$ or $B^n(f; P) \geqslant B^{n+1}(f; P)$ holds for all positive integer n and all points of T, then f cannot have a strict local maximum at an interior point of T.

1. INTRODUCTION

Let $B^n(f; x)$ $(n \geqslant 1)$ be the nth Bernstein polynomial of a real function $f(x)$ defined in $[0, 1]$:

$$B^n(f; x) := \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_i^n(x), \tag{1.1}$$

where

$$B_i^n(x) := \binom{n}{i} x^i (1-x)^{n-i}, \tag{1.2}$$

$i = 0, 1, 2, ..., n$. It is well known that, for a function $f(x)$ convex in $[0, 1]$,

$$B^n(f; x) \geqslant B^{n+1}(f; x) \tag{1.3}$$

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for all $n \geqslant 1$ and all $x \in [0, 1]$ (by B. Averbach, see [2, p. 115]) and hence, by the convergence of B_n to f,

$$B^n(f; x) \geqslant f(x) \tag{1.4}$$

for all $n \geqslant 1$ and all $x \in [0, 1]$. These results have been extended to include a class of approximation formulas (by S. Karlin, see [5], for example).

Conversely, it has been shown by Kosmak [3] that the condition (1.3) suffices to ensure the convexity of a twice continuously differentiable function. Furthermore, it has been proved that if $f(x)$ is continuous in $[0, 1]$ and the inequality (1.4) persists for all n, then $f(x)$ is convex (see [5], for example). It is obvious that the last theorem implies Kosmaks theorem by the convergence of the Bernstein approximation to f. These two results are called the converse theorems of convexity and have been extended to quite a wide class of positive linear operators by Ziegler [5].

Efforts to extend all these results to multivariate Bernstein polynomials were made a couple of years ago. Given is a triangle T with vertices T_1, T_2, and T_3, which will be called the domain triangle. A point P in T, which has the barycentric coordinates (u, v, w) with respect to T, will be written as $P = (u, v, w)$ in which nonnegative real numbers u, v, and w satisfy $u + v + w = 1$. Let $f(P)$ be a function defined on T. The nth Bernstein polynomial of $f(P)$ over the domain triangle is given by

$$B^n(f; P) := \sum_{i+j+k=n} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right) B^n_{i,j,k}(P), \tag{1.5}$$

where

$$B^n_{i,j,k}(P) := \frac{n!}{i!\,j!\,k!} u^i v^j w^k, \tag{1.6}$$

in which the nonnegative integers i, j, and k satisfy $i + j + k = n$. If $f(P)$ is continuous over T, written briefly as $f \in C(T)$, then

$$\lim_{n \to \infty} B^n(f; P) = f(P) \tag{1.7}$$

uniformly on T (see [4]).

For a convex function $f(P)$ on T, Chang and Davis prove in [1] that the sequence of the Bernstein polynomials is still decreasing as n goes to infinity. This result stimulated the present authors to find some converse theorems. It is impossible to have direct extensions of the converse theorems of convexity for the univariate Bernstein polynomials to the triangular case, as we have the following simple counter-example. Consider the standard triangle T with vertices $(0, 0)$, $(1,0)$, and $(0, 1)$ in the Cartesian plane. Let

$$f(x, y) := -xy,$$

its nth Bernstein polynomial is

$$B^n(f; x, y) = -\left(1 - \frac{1}{n}\right) xy.$$

Although the sequence of the Bernstein polynomials is decreasing and the inequality

$$B^n(f; x, y) \geqslant f(x, y)$$

holds for all n and all (x, y) in the standard triangle T, the function $-xy$ is not convex in T as its Gaussian curvature is always negative. Note that this function f does not attain local maximum inside the triangle T.

In the present paper, we prove the following two converse theorems of the convexity for the triangular Bernstein polynomials:

THEOREM 1. *If $f \in C(T)$ and the inequality*

$$B^n(f; P) \geqslant B^{n+1}(f; P) \tag{1.8}$$

holds for all natural numbers n and all points on T, then the function $f(P)$ does not attain strict local maximum inside the domain triangle.

THEOREM 2. *If $f \in C(T)$ and the inequality*

$$B^n(f; P) \geqslant f(P) \tag{1.9}$$

holds for all natural numbers n and all points on T, then the function $f(P)$ does not attain strict local maximum inside the domain triangle.

By a strict local maximum $f(P_0)$, we mean that f attains a local maximum at P_0 and is not a constant in any neighborhood of P_0.

It is clear that Theorem 1 can be deduced from Theorem 2 by the convergence (1.7) for a continuous approximated function f. Hence it suffices to prove Theorem 2. As a simple consequence of Theorem 1 or Theorem 2, we shall show that if $f(P)$ is a continuous and piecewise linear with respect to a triangulation of T, then each of (1.8) and (1.9) implies the convexity of $f(P)$ over T. Finally, we point out that the aforementioned converse theorems of univariate Bernstein polynomials can be derived from the present two theorems.

2. Lemmas

We begin with some definition. Points $(i/n, j/n, k/n)$, in which $i+j+k=n$, are called the nodes of the nth partition $S_n(T)$ of the domain triangle T, and $S_n(T)$ consists of n^2 subtriangles, each of them has three closest nodes as its vertices. The partition $S_4(T)$ is illustrated in Fig. 1.

Let $Q := (p, q, r)$ be an arbitrarily given point inside the domain triangle, i.e., p, q, r are positive with $p+q+r=1$. Let $B^n_{i,j,k} := B^n_{i,j,k}(Q)$ for brevity. We assign each value $B^n_{i,j,k}$ to the corresponding node $(i/n, j/n, k/n)$ of $S_n(T)$. Six lemmas will be presented in this section. The first five of them describe the value distribution of $(n+1)(n+2)/2$ real numbers $B^n_{i,j,k}$ with respect to the assignment.

Set

$$\alpha := \max(p/q, q/p, p/r, r/p, q/r, r/q),$$
$$\beta := \min(p/q, q/p, p/r, r/p, q/r, r/q).$$

It is easy to show that

$$\frac{B^n_{i+1,j-1,k}}{B^n_{i,j,k}} = \frac{jp}{q(i+1)} \qquad (i+j=n-k). \tag{2.1}$$

From (2.1) it follows immediately that

Lemma 1. *We have*

$$\begin{aligned} B^n_{i,j,k} &\leqslant B^n_{i+1,j-1,k}, \qquad \text{if} \quad (i+1)/j \leqslant p/q; \\ B^n_{i,j,k} &\geqslant B^n_{i+1,j-1,k}, \qquad \text{if} \quad (i+1)/j \geqslant p/q. \end{aligned} \tag{2.2}$$

Lemma 2. *For $\delta \in (0, p/q)$, we have*

$$B^n_{i,j,k} < (1+\beta\delta)^{-1} B^n_{i+1,j-1,k} \tag{2.3}$$

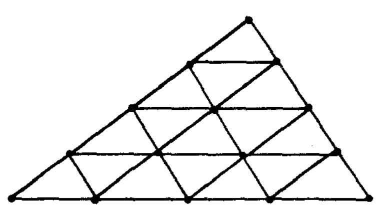

Figure 1

if $(i+1)/j \leqslant (p/q-\delta)$; *and*

$$B^n_{i,j,k} \geqslant (1+\beta\delta)\, B^n_{i+1,j-1,k} \tag{2.4}$$

if $(i+1)/j \geqslant (p/q+\delta)$.

Proof. Assume that $(i+1)/j \leqslant (p/q-\delta)$, by (2.1) then we have

$$\begin{aligned}\frac{B^n_{i+1,j-1,k}}{B^n_{i,j,k}} &= \frac{j}{i+1}\frac{p}{q} \geqslant \left(\frac{p}{q}-\delta\right)^{-1}\frac{p}{q}\\ &= 1+\delta\left(\frac{p}{q}-\delta\right)^{-1} > 1+\delta q/p \geqslant 1+\beta\delta.\end{aligned}$$

Similarly, if $(i+1)/j \geqslant (p/q+\delta)$, then

$$\begin{aligned}\frac{B^n_{i+1,j-1,k}}{B^n_{i,j,k}} &\leqslant (p/q+\delta)^{-1}\, p/q = (1+\delta q/p)^{-1}\\ &\leqslant (1+\beta\delta)^{-1}.\end{aligned}$$

By permuting (i,j,k) and (p,q,r) in the same manner in (2.2), (2.3), and (2.4) simultaneously, other inequalities can be obtained.

LEMMA 3. (1) *For* $k \in (0, 1, 2, ..., n)$, *we have*

$$\sum_{i+j=n-k} B^n_{i,j,k} = B^n_k(r); \tag{2.5}$$

(2) *if* $k \leqslant (n+1)(r-\delta)-1$ *and* $0<\delta<r$, *then*

$$B^n_k(r) < (1+\delta)^{-1}\, B^n_{k+1}(r); \tag{2.6}$$

(3) *if* $k_0 \leqslant (n+1)(r-\delta)-1$, *then*

$$\sum_{\substack{i+j+k=n\\ k\leqslant k_0}} B^n_{i,j,k} < (1+\delta)\,\delta^{-1} B^n_{k_0}(r). \tag{2.7}$$

Proof. The equality (2.5) comes from straightforward calculation. For the proof of (2.6), note that

$$\begin{aligned}\frac{B^n_{k+1}(r)}{B^n_k(r)} &= \frac{n-k}{k+1}\frac{r}{1-r} = \left(1-\frac{k+1}{n+1}\right)\left(\frac{k+1}{n+1}\right)^{-1}\frac{r}{1-r}\\ &\geqslant \frac{(1-r+\delta)\,r}{(r-\delta)(1-r)} > 1+\delta.\end{aligned}$$

The condition for (2.6) can be rewritten as $(k+1)/(n+1) \leqslant r-\delta$. By using (2.6) repeatedly, we obtain

$$B^n_{k_0-s}(r) < (1+\delta)^{-s} B^n_{k_0}(r),$$

where $s = 1, 2, 3, \ldots$ By (2.5) we get

$$\sum_{\substack{i+j+k=n \\ k \leqslant k_0}} B^n_{i,j,k} = \sum_{k=0}^{k_0} B^n_k(r)$$

$$< B^n_{k_0}(r) \sum_{k=0}^{\infty} (1+\delta)^{-k} = (1+\delta)\,\delta^{-1} B^n_{k_0}(r).$$

Lemma 4. *For arbitrarily fixed* $\varepsilon \in (0, 1)$, *assume that*

$$n \geqslant \varepsilon^{-1}(p^{-1} + q^{-1} + r^{-1}).$$

Let $B^n_{i^*,j^*,k^*}$ *be the maximum of all* $B^n_{i,j,k}$ *with* $i+j+k=n$, *we must have*

$$i^*/n > p(1-\varepsilon), \qquad j^*/n > q(1-\varepsilon), \qquad k^*/n > r(1-\varepsilon). \tag{2.8}$$

Proof. Suppose that at least one of (2.8) does not hold for a triple (i, j, k), we shall show that the corresponding $B^n_{i,j,k}$ is not the maximum. Without loss of generality, say that $i \leqslant np(1-\varepsilon)$. Then we have

$$np(1-\varepsilon) + j + k \geqslant i + j + k = n(p+q+r),$$

hence

$$j + k \geqslant n(p\varepsilon + q + r). \tag{2.9}$$

From (2.9) we conclude that among the following two inequalities

$$j \geqslant n(q + \varepsilon p/2) \qquad \text{and} \qquad k \geqslant n(r + \varepsilon p/2),$$

at least one of them is true. Say, for example, the first one holds, then we have

$$j \geqslant n(q + \varepsilon p/2) > nq > 1. \tag{2.10}$$

From (2.10) we know that $B^n_{i+1,j-1,k}$ makes sense.

On the other hand, we have $n > (\varepsilon p)^{-1}$, thus

$$i + 1 < np(1-\varepsilon) + np\varepsilon = np. \tag{2.11}$$

The combination of (2.10) and (2.11) gives $(i+1)/j < p/q$, hence we get by Lemma 1

$$B^n_{i,j,k} < B^n_{i+1,j-1,k},$$

this means that $B^n_{i,j,k}$ is not the maximum. This completes the proof of Lemma 4.

For $\varepsilon \in (0, 1)$, we define

$$\Omega_\varepsilon := \{(u, v, w): u \geqslant p(1-\varepsilon), v \geqslant q(1-\varepsilon), w \geqslant r(1-\varepsilon), u+v+w=1\}.$$

This is a closed triangle contained by the domain triangle T and containing the point Q as its interior point. Each side of Ω_ε is parallel to the corresponding side of T. It is clear that for $0<\delta<\varepsilon<1$, Ω_δ is contained by Ω_ε. It is reasonable to define that $\Omega_0 = Q$ and that $\Omega_1 = T$ (Fig. 2).

The Lemma 4 can be restated geometrically as follows. For $\varepsilon \in (0, 1)$ and $n \geqslant (1/p + 1/q + 1/r)/\varepsilon$, if $B^n_{i,j,k}(Q)$ is the largest, then we must have $(i/n, j/n, k/n) \in \Omega_\varepsilon$.

LEMMA 5. *For any $\varepsilon \in (0, 1)$ there exists $\delta \in (0, \varepsilon)$ and a positive integer n_0 such that for $n \geqslant n_0$, if $(i/n, j/n, k/n) \bar{\in} \Omega_\varepsilon$ and $(i_0/n, j_0/n, k_0/n) \in \Omega_\delta$, then*

$$B^n_{i,j,k}(Q) < B^n_{i_0,j_0,k_0}(Q).$$

Proof. For a given $\varepsilon \in (0, 1)$, it is always possible to find a $\delta \in (0, \varepsilon)$ such that

$$\frac{1-\delta}{1+2\delta\alpha} > \left(1 + \frac{\beta^2\varepsilon}{2}\right)^{-\sigma\varepsilon}, \tag{2.12}$$

where

$$\sigma := (p^{-1} + q^{-1} + r^{-1})^{-1}/4. \tag{2.13}$$

Then we take n_0 sufficiently large such that

$$\sigma\, \delta n_0 > 2. \tag{2.14}$$

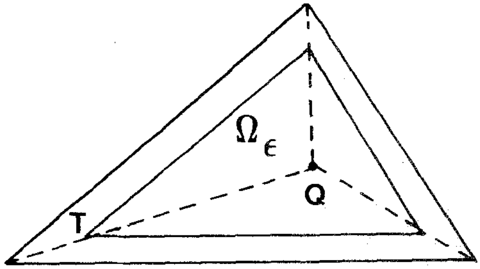

FIGURE 2

We are going to show that such δ and n_0 can meet our requirement. Let $n \geqslant n_0$ and $(i/n, j/n, k/n) \bar{\in} \Omega_\varepsilon$. Without loss of generality, we may assume that

$$i < np(1-\varepsilon). \tag{2.15}$$

Just as in the proof of Lemma 4, we can assume that

$$j \geqslant n(q + p\varepsilon/2). \tag{2.16}$$

Consider the following sequence

$$B^n_{i,j,k},\ B^n_{i+1,j-1,k},\ \dots,\ B^n_{i+s_0,j-s_0,k},$$

where

$$s_0 := \max\{s\colon i+s < np(1-\varepsilon/2),\ j-s > nq\}.$$

It is clear that $s_0 = \min(s_1, s_2)$, where

$$\begin{aligned} s_1 &:= \max\{s\colon i+s < np(1-\varepsilon/2)\}, \\ s_2 &:= \max\{s\colon j-s > nq\}. \end{aligned}$$

By the definition of s_1 and (2.15), we get

$$s_1 \geqslant np(1-\varepsilon/2) - i - 1 > np(1-\varepsilon/2) - 1 - np(1-\varepsilon),$$

so that

$$s_1 > np\varepsilon/2 - 1. \tag{2.17}$$

Similarly, by the definition of s_2 and (2.16), we have

$$s_2 \geqslant j - nq - 1 \geqslant n(q + p\varepsilon/2) - nq - 1,$$

hence

$$s_2 \geqslant np\varepsilon/2 - 1. \tag{2.18}$$

From (2.17) and (2.18) it follows that

$$s_0 = \min\{s_1, s_2\} \geqslant np\varepsilon/2 - 1.$$

From the definition of s_0 we see that

$$\frac{i+s_0}{j-s_0} < \frac{p}{q}\left(1 - \frac{\varepsilon}{2}\right),$$

hence

$$\frac{i+s+1}{j-s} < \frac{p}{q} - \frac{p\varepsilon}{2q} < \frac{p}{q} - \frac{\varepsilon\beta}{2}, \tag{2.19}$$

for $s = 0, 1, ..., s_0 - 1$. Comparing (2.19) with the first result of Lemma 2, we obtain

$$B^n_{i+s, j-s, k} < B^n_{i+s+1, j-s-1, k}(1 + \varepsilon\beta^2/2)^{-1}.$$

Inductively, we get

$$\begin{aligned} B^n_{i,j,k} &< (1 + \varepsilon\beta^2/2)^{-s_0} B^n_{i+s_0, j-s_0, k} \\ &\leqslant (1 + \varepsilon\beta^2/2)^{-(np\varepsilon/2 - 1)} B^n_{i^*, j^*, k^*}, \end{aligned}$$

in which $B^n_{i^*, j^*, k^*}$ denotes the maximum of all $B^n_{i,j,k}$. Since $p > 4\sigma$ by (2.13) and $n\sigma\varepsilon > n_0\sigma\delta > 2$ by (2.14), we have

$$np\varepsilon/2 - 1 > n\sigma\varepsilon,$$

therefore

$$B^n_{i,j,k} < (1 + \varepsilon\beta^2/2)^{-n\sigma\varepsilon} B^n_{i^*, j^*, k^*}. \tag{2.20}$$

We shall point out that if $n \geqslant n_0$ then

$$(i^*/n, j^*/n, k^*/n) \in \Omega_\delta.$$

In fact, by (2.13) and (2.14) it follows that

$$n_0 > (1/p + 1/q + 1/r)\, 8/\delta,$$

hence if $n \geqslant n_0$ we conclude by Lemma 4 that

$$(i^*/n, j^*/n, k^*/n) \in \Omega_{\delta/8} \subset \Omega_\delta.$$

Let $(i/n, j/n, k/n)$ be any point in Ω_δ, so that

$$i \geqslant np(1-\delta), \qquad j \geqslant nq(1-\delta), \qquad k \geqslant nr(1-\delta).$$

Since

$$n(p+q+r) = i + j + k \geqslant i + nq(1-\delta) + nr(1-\delta),$$

we have immediately

$$np(1-\delta) \leqslant i \leqslant n[p + (q+r)\,\delta],$$

equivalently

$$np(1-\delta) \leqslant i \leqslant n[p(1-\delta)+\delta]. \tag{2.21}$$

Note that there are similar inequalities for j and k. If $(i'/n, j'/n, k'/n)$ is another point in Ω_δ, then by (2.21) we have

$$|i-i'| \leqslant n\delta. \tag{2.22}$$

Consider two adjacent nodes in the triangle Ω_δ. Without loss of generality, say $(i/n, j/n, k/n)$ and $((i+1)/n, (j-1)/n, k/n)$. The ratio of the two Bernstein basis polynomials associated with these two points is given by (2.1), namely $jp/(i+1)\,q$, which is less than

$$\frac{jp}{iq} \leqslant \frac{n[q+(p+r)\,\delta]\,p}{np(1-\delta)\,q} \leqslant \frac{1+2\delta\alpha}{1-\delta}.$$

It gives

$$B^n_{i+1,j-1,k} < \left(\frac{1+2\delta\alpha}{1-\delta}\right) B^n_{i,j,k}. \tag{2.23}$$

Any two nodes of $S_n(T)$ can always be connected by a broken line formed by joining two neighboring nodes with line segment, and the broken line passes through at most $n+1$ nodes. Keep this fact in mind, we know that if a point $(i_0/n, j_0/n, k_0/n)$ is in Ω_δ, by using (2.23) repeatedly and by (2.22), then we obtain

$$B^n_{i^*,j^*,k^*} \leqslant \left(\frac{1+2\delta\alpha}{1-\delta}\right)^{n\delta} B^n_{i_0,j_0,k_0}. \tag{2.24}$$

Combining (2.20) and (2.24), we get

$$B^n_{i,j,k} < \left(\frac{1+2\delta\alpha}{1-\delta}\right)^{n\delta} \left(1+\frac{\beta^2\varepsilon}{2}\right)^{-n\sigma\varepsilon} B^n_{i_0,j_0,k_0}. \tag{2.25}$$

In virtue of (2.12), it comes from (2.25) that

$$B^n_{i,j,k} < B^n_{i_0,j_0,k_0}. \tag{2.26}$$

The proof of Lemma 5 is completed.

LEMMA 6. *Let $\Delta_d \subset T$ be a triangle with sides parallel to that of the domain triangle T, and*

$$d := \frac{\operatorname{area}(\Delta_d)}{\operatorname{area}(T)},$$

$N_n(d) :=$ the number of nodes of $S_n(T)$ which belong to Δ_d. There exists a positive integer n_0 such that if $n \geqslant n_0$ then

$$N_n(d) > dn^2/4. \tag{2.27}$$

Proof. Note that the number of nodes in $S_n(T)$ is $(n+1)(n+2)/2$. The desired result comes from the simple fact

$$\lim_{n \to \infty} \frac{2N_n(d)}{(n+1)(n+2)} = d > d/2 > 0.$$

3. Proof of Theorem 2

Suppose, contrarily, the continuous function f attains a strict local maximum at a point $Q = (p, q, r)$ interior to the domain triangle. Without loss of generality, we assume that $f(Q) = 0$, hence there exists $\varepsilon \in (0, 1)$ such that f is nonpositive on the triangular region Ω_ε. By Lemma 5, there are a positive integer n_1 and $\delta \in (0, \varepsilon)$ such that if $n \geqslant n_1$, $(i/n, j/n, k/n) \notin \Omega_\varepsilon$, and $(i_0/n, j_0/n, k_0/n) \in \Omega_\delta$ then

$$B^n_{i,j,k} < B^n_{i_0, j_0, k_0}. \tag{3.1}$$

Note that all these Bernstein basis polynomials are evaluated at the point Q. The "strictness" of the local maximum insures that there is a triangular region Δ_d ($\subset \Omega_\delta$) which has sides parallel to that of T, on which the supremum of f, denoted by $(-h)$, shall be negative. By Lemma 6, there exists a positive integer n_2 such that the number of the nodes (of $S_n(T)$) which belong to Δ_d will be greater than $n^2 d/4$ for $n \geqslant n_2$. Let

$$L := \operatorname*{supremum}_{P \in T} |f(P)|.$$

Split the sum in (1.5) into the following three parts

$$\Sigma_1 + \Sigma_2 + \Sigma_3,$$

in which the first, second, and third are the summations over the nodes outside Ω_ε, inside Δ_d, and on $\Omega_\varepsilon \backslash \Delta_d$, respectively. Let

$$a_n := \max\{B^n_{i,j,k} \colon (i/n, j/n, k/n) \notin \Omega_\varepsilon\},$$
$$b_n := \min\{B^n_{i,j,k} \colon (i/n, j/n, k/n) \in \Omega_\delta\}.$$

From (3.1) we know that $a_n < b_n$ for $n \geqslant n_1$. By the definition of Ω_ε, it is clear that

$$\sum_{(i/n,\, j/n,\, k/n) \bar{\in} \Omega_\varepsilon} B^n_{i,j,k} \leqslant \sum_{i < np(1-\varepsilon)} + \sum_{j < nq(1-\varepsilon)} + \sum_{k < nr(1-\varepsilon)} \tag{3.2}$$

The first summation on the right-hand side of (3.2) can be rewritten as

$$\sum_{\substack{i+j+k=n,\\ i \leqslant i_0}} B^n_{i,j,k},$$

where

$$i_0 := \max\{i : i < np(1-\varepsilon)\}.$$

Let

$$K := \max\left\{\frac{2}{p\varepsilon}, \frac{2}{q\varepsilon}, \frac{2}{r\varepsilon}\right\}.$$

If $n \geqslant K$, then $i_0 < np(1-\varepsilon) < (n+1)\, p(1-\varepsilon) = (n+1)(p - p\varepsilon/2) - (n+1)\, p\varepsilon/2$, thus $i_0 < (p - p\varepsilon/2)(n+1) - 1$. By Lemma 3, we obtain

$$\sum_{i < np(1-\varepsilon)} B^n_{i,j,k} \leqslant [1 + 2/(p\varepsilon)] \sum_{j+k=n-i_0} B^n_{i,j,k} \leqslant 2n(1+K)\, a_n;$$

the same upper bound applies to the second and the third summation in (3.2). Hence

$$\sum_{(i/n,\, j/n,\, k/n) \bar{\in} \Omega_\varepsilon} B^n_{i,j,k} \leqslant 6n(1+K)\, a_n,$$

so that

$$|\Sigma_1| \leqslant 6n(1+K)\, La_n. \tag{3.3}$$

It is clear that

$$\Sigma_3 \leqslant 0. \tag{3.4}$$

Furthermore, since

$$\sum_{(i/n,\, j/n,\, k/n) \in \Delta_d} B^n_{i,j,k} \geqslant n^2 d b_n/4,$$

we see that

$$\Sigma_2 < -n^2 d b_n h/4. \tag{3.5}$$

From (3.3), (3.4), and (3.5) it follows that

$$\begin{aligned} B^n(f; Q) &< |\Sigma_1| + \Sigma_2 \\ &\leqslant 6L(1+K)\, na_n - n^2 db_n h/4 < nb_n[6(1+K)\, L - ndh/4]. \end{aligned}$$

If $n \geqslant \max\{n_1, n_2, K, 24(1+K)\, L/(dh)\}$, then $B^n(f; Q) < 0$, an impossibility. The proof of Theorem 2 is completed.

4. Corollaries

Corollary 1. *Let $f(P)$ be continuous on domain triangle T, and piecewise linear with respect to a finite triangulation of T. If f satisfies the inequality*

$$B^n(f; P) \geqslant f(P) \tag{4.1}$$

or

$$B^n(f; P) \geqslant B^{n+1}(f; P), \tag{4.2}$$

for all positive integer n and all $P \in T$, then f must be convex over T.

Proof. Since the Bernstein operator reproduces linear functions, if $f(P)$ satisfies (4.1) or (4.2) then so does $f(P) + g(P)$, where $g(P)$ is a linear function. Suppose, contrarily, f is not convex over T; a linear function g can be found such that $f + g$ assumes a strict local maximum at some point interior to T, a contradiction by Theorem 1 and Theorem 2.

It is obvious that Theorem 1 and Theorem 2 are still valid for univariate case, i.e., the Bernstein polynomials. From the univariate versions, we can derive the two well-known results of Bernstein polynomials mentioned already in the Introduction, we present them here as

Corollary 2. *Let $f(x)$ be a continuous function on $[0, 1]$. If $f(x)$ satisfies*

$$B^n(f; x) \geqslant f(x) \tag{4.3}$$

or

$$B^n(f; x) \geqslant B^{n+1}(f; x) \tag{4.4}$$

for all positive integer n and all $x \in [0, 1]$, then $f(x)$ must be convex in this interval.

Proof. If f is not convex in the interval, there exist two points $u < y$ in $(0, 1)$ such that

$$f[(u+y)/2] > [f(u)+f(y)]/2.$$

Let

$$g(x) := f(u)(y-x)/(y-u) + f(y)(x-u)/(y-u)$$

and consider

$$F(x) := f(x) - g(x).$$

It is clear that $F(u) = F(y) = 0$ and

$$F[(u+y)/2] > 0.$$

Let m be the maximum of $F(x)$ on the interval $[u, y]$, hence we have $m > 0$. Suppose that v is the leftmost point in (u, y) such that $F(v) = m$, the existence of v follows from the continuity of F. We now see that F assumes a strict local maximum at the point v. This will lead to a contradiction in the same way as we saw in the proof of Corollary 1.

The contribution of this paper has been extended to Bernstein polynomials over higher dimensional simplices by Yang Lu and the present authors. A paper had been submitted for publication in 1987. When we revised the present paper, we received a preprint by Wolfgang Dahmen and Charles A. Michelli, titled "Convexity and Bernstein Polynomials on k-Simploids." They get a weaker extension based on semigroup techniques and the maximum principle for second order elliptic operators, a quite different approach from ours.

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References

1. G. Z. Chang and P. J. Davis, The convexity of Bernstein polynomials over triangles, *J. Approx. Theory* **40** (1984), 11–28.
2. P. J. Davis, "Interpolation and Approximation," Dover, New York, 1975.
3. L. Kosmak, A note on Bernstein polynomials of a convex function, *Math. (Cluj)* **2** (1960), 281–282.
4. G. G. Lorentz, "Bernstein Polynomials," Chelsea, New York, 1986.
5. Z. Ziegler, Linear approximation and generalized convexity, *J. Approx. Theory* **1** (1968), 420–433.